A posteriori error estimates of the discontinuous Galerkin method for linear elliptic and parabolic problems

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A posteriori error estimates

- can be extracted from the discrete solution and given data of the problem
- u - a weak solution of the problem, u_h - its discrete solution
  usual form:
  $$\|u - u_h\| \le c f(u_h), \tag{1}$$
  where c is a constant and f is a function of the discrete solution

Discontinuous Galerkin method

- methods for stationary problems
  - Galerkin orthogonality principle
  - Helmholtz decomposition
  - duality principle

- nonstationary problem
  - Helmholtz decomposition

Poisson's equation - formulation

Let $\Omega \in \mathbb{R}^d$ (d=2 or 3) be a bounded polyhedral domain with a boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\partial\Omega_D \cap \partial\Omega_N = \emptyset$. Let us consider the problem:

$$\begin{aligned} -\triangle u &= f && \text{in} \quad \Omega, \\ u &= g_D && \text{on} \quad \partial\Omega_D, \\ \nabla u \cdot n &= g_N && \text{on} \quad \partial\Omega_N, \end{aligned} \tag{2}$$

where n denote the outward unit normal vector to $\partial\Omega$, $g_D \in H^{1/2}(\partial\Omega_D)$ and $g_N \in H^{-1/2}(\partial\Omega_N)$. Let $f \in L^2(\Omega)$.

Poisson's equation - notation

- $\mathcal{T}_h$, $h > 0$: a family of partitions of Ω into a finite number of closed triangles in 2D and tetrahedra in 3D with mutually disjoint interiors
- ρ_K: the radius of the largest d-dimensional ball inscribed into K
- $h_K =$ diam(K)
- $\mathcal{F}_h^I$, $\mathcal{F}_h^D$ and $\mathcal{F}_h^N$ denote the set of all interior edges, edges on $\partial\Omega_D$ and edges on $\partial\Omega_N$, respectively.
- $\mathcal{F}_h^{DN} \equiv \mathcal{F}_h^D \cup \mathcal{F}_h^N$, $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^D \cup \mathcal{F}_h^N$
- $\forall\ \Gamma \in \partial\Omega$: either $\Gamma \in \mathcal{F}_h^D$, or $\Gamma \in \mathcal{F}_h^N$
- $h_\Gamma =$ diam(Γ)

Poisson's equation - notation

- n_Γ: a unit normal vector to $\Gamma \in \mathcal{F}_h$
- $\forall\, \Gamma \in \mathcal{F}_h^I$: K_Γ^L and K_Γ^R denote elements, which share this edge, the orientation of n_Γ: pointed out of K_Γ^L
- $\forall\, \Gamma \in \mathcal{F}_h^{DN}$: the same orientation as the outward normal to $\partial\Omega$

$$H^s(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^s(K)\, \forall K \in \mathcal{T}_h\}, \tag{3}$$

$$\|v\|^2_{H^s(\Omega,\mathcal{T}_h)} = \sum_{K\in\mathcal{T}_h} \|v\|^2_{H^s(K)}, \tag{4}$$

$$S_{hp} = \{v; v \in L^2(\Omega), v|_K \in P^p(K)\, \forall\, K \in \mathcal{T}_h\}, \tag{5}$$

Triangulation - assumptions

- shape regularity:

$$\exists C_s > 0: \; \frac{h_K}{\rho_K} \leq C_s \; \forall \, K \in \mathcal{T}_h, \tag{6}$$

- local quasi-uniformity:

$$\begin{gathered} \exists C_H > 0: \; h_K \leq C_H h_{K'} \\ \forall \, K, \; K^{'}: \partial K \cap \partial K^{'} \neq \emptyset. \end{gathered} \tag{7}$$

Poissonʼs equation - notation

- For $v \in H^1(\Omega, \mathcal{T}_h)$ we denote:

$$v_\Gamma^L = \text{the trace of } v|_{K_\Gamma^L} \text{ on } \Gamma,\ \Gamma \in \mathcal{F}_h^I, \tag{8}$$

$$v_\Gamma^R = \text{the trace of } v|_{K_\Gamma^R} \text{ on } \Gamma,\ \Gamma \in \mathcal{F}_h^I, \tag{9}$$

$$\langle v \rangle_\Gamma = \frac{1}{2}(v_\Gamma^L + v_\Gamma^R),\ \Gamma \in \mathcal{F}_h^I, \tag{10}$$

$$[v]_\Gamma = v_\Gamma^L - v_\Gamma^R,\ \Gamma \in \mathcal{F}_h^I, \tag{11}$$

$$v_\Gamma^L = \text{the trace of } v|_{K_\Gamma^L} \text{ on } \Gamma,\ \Gamma \in \mathcal{F}_h^{DN}, \tag{12}$$

$$\langle v \rangle_\Gamma = [v]_\Gamma = v_\Gamma^L,\ \Gamma \in \mathcal{F}_h^{DN}, \tag{13}$$

Discretization

$$a_h^k(u,v) = \sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla v \, dx$$
$$- \sum_{\Gamma \in \mathcal{F}_h^{ID}} \int_\Gamma \Big(\langle \nabla u \cdot n \rangle [v] - \theta \langle \nabla v \cdot n \rangle [u] \Big) \, dS, \qquad (14)$$

$$F_h^k(v) = \int_\Omega f v \, dx + \sum_{\Gamma \in \mathcal{F}_h^N} \int_\Gamma g_N v \, dS + \theta \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma (\nabla v \cdot n) g_D \, dS, \qquad (15)$$

where $k \in \{S, N, I\}$, $\theta = -1$ is connected with the symetric form, $\theta = 1$ the nonsymetric form and $\theta = 0$ the incomplete form of the discontinuous Galerkin method.

Discretization

$$J_h^\sigma(u,v) = \sum_{\Gamma\in\mathcal{F}_h^{ID}} \int_\Gamma \sigma[u][v]\, dS, \tag{16}$$

$$J_D^\sigma(v) = \sum_{\Gamma\in\mathcal{F}_h^{D}} \int_\Gamma \sigma g_D v\, dS, \tag{17}$$

$$\sigma|_\Gamma = \frac{C_W}{\max\{h_{K_\Gamma^L}, h_{K_\Gamma^R}\}} \quad \text{for} \quad \Gamma \in \mathcal{F}_h^I, \tag{18}$$

$$\sigma|_\Gamma = \frac{C_W}{h_{K_\Gamma^L}} \quad \text{for} \quad \Gamma \in \mathcal{F}_h^D, \tag{19}$$

where C_W is a suitable constant ensuring coercivity of $\mathcal{B}_h^{k,\sigma}$.

Discretization

$$\mathcal{B}_h^{k,\sigma}(u,v) = a_h^k(u,v) + J_h^\sigma(u,v), \quad k \in \{S,N,I\}, \tag{20}$$

$$l_h^{k,\sigma}(v) = F_h^k(v) + J_D^\sigma(v), \quad k \in \{S,N,I\}. \tag{21}$$

Definition 1 *Function u_h is called a discontinuous Galerkin approximation of the solution of the problem* (2)*, if it is the solution of one of the following problems: Find $u_h \in S_{hp}$ such that*

$$\mathcal{B}_h^{k,\sigma}(u_h, v_h) = l_h^{k,\sigma}(v_h) \quad \forall\, v_h \in S_{hp}, \tag{22}$$

where $k \in \{S,N,I\}$.

Theorem 1 ***(Multiplicative trace inequality)*** *There exists a constant $C_M > 0$ independent of v, h and K such that*

$$\|v\|^2_{\partial K} \leq C_M(\|v\|_K|v|_{1,K} + h_K^{-1}\|v\|^2_K), \quad K \in \mathcal{T}_h,\ v \in H^1(K). \tag{23}$$

Theorem 2 ***(Inverse inequality)*** *There exists a constant $C_I > 0$ independent of v, h and K such that*

$$|v|_{1,K} \leq C_I h_K^{-1}\|v\|_K, \quad K \in \mathcal{T}_h,\ v \in P^p(K). \tag{24}$$

A posteriori error estimate

Definition 2 *Let u be the weak solution of* (2) *and u_h be its discontinuous Galerkin approximation. Let set*

$$e = u - u_h. \tag{25}$$

- the so-called DG-norm:

$$|||v||| \equiv \sum_{K \in \mathcal{T}_h} \int_K \nabla v \cdot \nabla v \, dx + \sum_{\Gamma \in \mathcal{F}_h^{ID}} \int_\Gamma \sigma [v][v] \, dS \quad \forall v \in H^1(\Omega, \mathcal{T}_h) \tag{26}$$

- L^2 norm

Definition 3 ***(Oswald's interpolation operator)*** *Let set* $V = H^1_{g_D,D}(\Omega) \equiv \{v \in H^1(\Omega); v = g_D \quad \text{on} \quad \partial\Omega_D\}$ *for mixed boundary conditions and* $V = H^1(\Omega)$ *for pure Neumann boundary conditions. Let* $\mathcal{N}^V$ *be the set of all Lagrangian nodes needed for construction of a function from* $S_{hp} \cap V$. *Oswald's interpolation operator* $\mathcal{I}^V_{Os} : \quad S_{hp} \to S_{hp} \cap V$ *depending on given boundary conditions is for* $v_h \in S_{hp}$ *defined by:*

$$\mathcal{I}^V_{Os}(v_h)(\nu) = \frac{1}{|\omega_\nu|} \sum_{K \in \omega_\nu} v_h|_K(\nu), \quad \nu \in \mathcal{N}^V \backslash \mathcal{N}^B_D \tag{27}$$

$$= g_D(\nu), \quad \nu \in \mathcal{N}^B_D \tag{28}$$

where $\omega_\nu = \{K \in \mathcal{T}_h;\ \nu \in K\}$, $\mathcal{N}^B_D = \{\nu \in \mathcal{N}^V; \nu \in \partial\Omega_D\}$.

Theorem 3 *Let $\mathcal{T}_h$ be conforming or nonconforming mesh consisting of triangles in 2D and tetrahedra in 3D. Assume that triangulation is regular and locally quasi-uniform. Let g_D be the restriction to $\partial\Omega_D$ of a function in $S_{hp} \cap H^1(\Omega)$. For any $v_h \in S_{hp}$, $i = 0, 1$ hold:*

$$\sum_{K \in \mathcal{T}_h} \|v_h - \mathcal{I}_{Os}^{V}(v_h)\|_{i,K}^2 \le C_{O2}^2 \left(\sum_{\Gamma \in \mathcal{F}_h^I} h_\Gamma^{1-2i} \|[v_h]\|_\Gamma^2 + \sum_{\Gamma \in \mathcal{F}_h^D} h_\Gamma^{1-2i} \|v_h - g_D\|_\Gamma^2 \right) \tag{29}$$

where C_{O2} is a constant independent of h and v_h.

Galerkin orthogonality of the error

Theorem 4 *There holds:*

$$\sum_{K\in\mathcal{T}_h}\|\nabla e\|_K^2 \le c\left(\sum_{K\in\mathcal{T}_h} h_K^2\|f+\triangle u_h\|_K^2 + \sum_{\Gamma\in\mathcal{F}_h^I} h_\Gamma\|[\partial_n u_h]\|_\Gamma^2\right.$$

$$+\sum_{\Gamma\in\mathcal{F}_h^N} h_\Gamma\|g_N-\partial_n u_h\|_\Gamma^2 + C_W^2\sum_{\Gamma\in\mathcal{F}_h^I} h_\Gamma^{-1}\|[u_h]\|_\Gamma^2$$

$$\left.+C_W^2\sum_{\Gamma\in\mathcal{F}_h^D} h_\Gamma^{-1}\|g_D-u_h\|_\Gamma^2\right), \tag{30}$$

where a constant c is independent of h and C_W.

Duality principle

- Ω: a convex domain
- Neumann's boundary condition on the whole boundary
- Let $\phi \in \{v; \frac{1}{|\Omega|}\int_\Omega v\,dx = 0\}$ be the solution of the dual problem:

$$\begin{aligned} -\triangle\phi &= e \quad \text{in} \quad \Omega, \\ \nabla\phi\cdot n &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{31}$$

and

$$\exists C > 0 : \quad \|\phi\|_{2,\Omega} \leq C\|e\|_\Omega. \tag{32}$$

- conforming, regular and locally quasi-uniform system of partitions $\{\mathcal{T}_h\}_{h>0}$ consisting of triangles in 2D and tetrahedra in 3D.

Duality principle

Theorem 5 *There holds:*

$$\|e\|_\Omega \leq c \left(\sum_{K \in \mathcal{T}_h} \|f + \triangle u_h\|_K^2 h_K^4 + \sum_{\Gamma \in \mathcal{F}_h^I} \|[\nabla u_h \cdot n]\|_\Gamma^2 h_\Gamma^3 \right.$$

$$+ \sum_{\Gamma \in \mathcal{F}_h^N} \|g_N - \nabla u_h \cdot n\|_\Gamma^2 h_\Gamma^3 + \sum_{\Gamma \in \mathcal{F}_h^I} \sigma^2 h_\Gamma^3 \|[u_h]\|_\Gamma^2$$

$$\left. + \sum_{\Gamma \in \mathcal{F}_h^I} h_\Gamma \|[u_h]\|_\Gamma^2 + 2 \sum_{\Gamma \in \mathcal{F}_h^I} h_\Gamma^{-1} \|[u_h]\|_\Gamma^2 \right)^{1/2}, \tag{33}$$

where a constant c is independent of h.

Helmholtz decomposition

Theorem 6 ***(About Helmholtz decomposition)*** *There exists decomposition*

$$\nabla_h e = \nabla \phi + \textit{curl}\, \chi, \tag{34}$$

where $\phi \in H^1_D(\Omega) \equiv \{v \in H^1(\Omega);\, v = 0 \quad \text{on} \quad \partial\Omega_D\}$ *is the solution of the problem*

$$\int_\Omega \nabla \phi \cdot \nabla v \, dx = \int_\Omega \nabla_h e \cdot \nabla v \, dx \quad \forall v \in H^1_D(\Omega), \tag{35}$$

$\chi \in H(\textit{curl}, \Omega)$ *and* $n \cdot \textit{curl}\, \chi = 0$ *on* $\partial\Omega_N$*.*

Helmholtz decomposition

Theorem 7 ***(Properties of the Helmholtz decomposition)*** *Helmholtz decomposition* (34) *is orthogonal in the sense that*

$$\|\nabla_h e\|_\Omega^2 = \|\nabla\phi\|_\Omega^2 + \|\textit{\textbf{curl}}\,\chi\|_\Omega^2. \qquad (36)$$

In addition, the estimate

$$\|\nabla\phi\|_\Omega + \|\textit{\textbf{curl}}\,\chi\|_\Omega \le 2|e|_{H^1(\Omega,\mathcal{T}_h)} \qquad (37)$$

holds.

Discrete normal flux

$$\Sigma_n(u_h) \equiv \begin{cases} \langle \partial_n(u_h) \rangle - \sigma[u_h], & \text{on} \quad \partial K \cap \mathcal{F}_h^I \\ \partial_n(u_h) - \sigma(u_h - g_D), & \text{on} \quad \partial K \cap \mathcal{F}_h^D \\ g_N, & \text{on} \quad \partial K \cap \mathcal{F}_h^N, \end{cases} \tag{38}$$

where

$$\sigma|_\Gamma = \frac{C_W}{\max\{h_{K_\Gamma^L}, h_{K_\Gamma^R}\}} \quad \text{for} \quad \Gamma \in \mathcal{F}_h^I \tag{39}$$

and

$$\sigma|_\Gamma = \frac{C_W}{h_{K_\Gamma^L}} \quad \text{for} \quad \Gamma \in \mathcal{F}_h^D, \tag{40}$$

respectively.

Discrete normal flux

- Properties:

$$\int_K f\,dx + \int_{\partial K} \Sigma_n(u_h) = 0 \quad \text{for} \quad \forall K \in \mathcal{T}_h, \tag{41}$$

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K\setminus\partial\Omega_D} n\cdot\nabla u\phi\,dS = \sum_{K\in\mathcal{T}_h} \int_{\partial K\setminus\partial\Omega_D} \Sigma_n(u_h)\phi\,dS. \tag{42}$$

Theorem 8 *There holds:*

$$\sum_{K\in\mathcal{T}_h} \|\nabla e\|_K^2 \le c \sum_{K\in\mathcal{T}_h} \Big(h_K^2 \|f + \triangle u_h\|_K^2$$
$$+ h_K \|\Sigma_n(u_h) - \partial_n u_h\|^2_{\partial K\cap\mathcal{F}_h^{IN}} + h_K^{-1} \|[u_h]\|^2_{\partial K\cap\mathcal{F}_h^{ID}} \Big) , \tag{43}$$

where c is independent of h.

A nonstationary problem

Let $\Omega \in \mathbb{R}^d$ (d=2 or 3) be a bounded polyhedral domain, $T > 0$ and $Q_T = \Omega \times (0, T)$. Let us consider the problem:

$$\begin{aligned} \frac{\partial u}{\partial t} - \triangle u &= f && \text{in} \quad Q_T, \\ u &= 0 && \text{on} \quad \partial\Omega \times (0, T), \\ u(x, 0) &= u^0(x) && \text{in} \quad \Omega. \end{aligned} \tag{44}$$

Assume that the data satisfy the following conditions:

$$\begin{aligned} & f \in C(0, T; H^{-1}(\Omega)), \\ & u^0 \in L^2(\Omega). \end{aligned} \tag{45}$$

Definition 4 *The weak solution of the problem* (44) *is defined as a function* $u \in L^2(0,T;H_0^1(\Omega))$ *satisfying the conditions:*

$$\begin{aligned} \langle \frac{\partial u(t)}{\partial t}, v\rangle + \int_\Omega \nabla u(t)\cdot\nabla v \, dx &= \langle f(t), v\rangle \\ \text{\textit{for}} \quad \forall v \in H_0^1(\Omega) \quad s.v. \quad t\in(0,T), & \\ u(x,0) &= u^0(x) \quad \text{\textit{in}} \quad \Omega, \end{aligned} \tag{46}$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality pairing between* $H_0^1(\Omega)$ *and* $H^{-1}(\Omega)$.

Time discretization

- A partition of $[0,T]$: $0 = t_0 < t_1 < ... < t_{\bar{N}} = T$
- Notation: $\tau_n = t_n - t_{n-1}$, $\tau = \max_{1\le n\le \bar{N}} \tau_n$
- The problem (46) is discretized in time by a backward Euler scheme:

Find a sequence $\{u^n\}_{1\le n\le \bar{N}}$, $u^n \in H_0^1(\Omega)$ such that

$$\int_\Omega \frac{u^n - u^{n-1}}{\tau_n} v \, dx + \int_\Omega \nabla u^n \cdot \nabla v \, dx = \int_\Omega f^n v \, dx \quad \text{for} \quad \forall v \in H_0^1(\Omega), \tag{47}$$

where $f^n \equiv f(\cdot, t_n)$.

Discretization in space

- discontinuous Galerkin methods (SIPG, NIPG, IIPG)
- On each time level is considered a system $\{\mathcal{T}_{hn}\}_{h>0}$ of partitions of Ω consisting of triangles in 2D and tetrahedra in 3D.
- The set of all interior edges and edges on boundary ($\partial\Omega_D = \partial\Omega$) is denoted by $\mathcal{F}_{hn}^I$ and $\mathcal{F}_{hn}^D$, respectively.
- Set $\mathcal{F}_{hn}^{ID} = \mathcal{F}_{hn}^I \cup \mathcal{F}_{hn}^D$, $h_n = \max\limits_{K\in\mathcal{T}_{hn}} h_K$
- The solution of the problem (47) is approximated by piecewise linear functions:

$$S_{h1}^n \equiv \{v; v \in L^2(\Omega), v|_K \in P^1(K) \, \forall\, K \in \mathcal{T}_{hn}\}. \qquad (48)$$

Triangulation - assumptions

Let triangulations $\{\mathcal{T}_{hn}\}_{h>0, 1\le n\le \bar{N}}$ be regular and locally quasi-uniform.

Assume that there exists a triangulation $\widetilde{\mathcal{T}}_{hn}$ satisfying (6) and (7) which is a refinement of both $\mathcal{T}_{hn-1}$ and $\mathcal{T}_{hn}$, $1 \le n \le \bar{N}$ and such that

$$\exists C_{HT} > 0 : \quad \sup_{1\le n\le \bar{N}} \sup_{K\in\widetilde{\mathcal{T}}_{hn}} \sup_{K'\in\mathcal{T}_{hn}, K\subset K'} : \ \frac{h_{K'}}{h_K} \quad < C_{HT}. \tag{49}$$

Full discretization

For a given approximation $u_h^0 \in S_{h1}^0$ of an initial condition u^0 find a sequence $\{u_h^n\}_{1\leq n\leq \bar{N}}$, $u_h^n \in S_{h1}^n$ such that

$$\int_\Omega \frac{u_h^n - u_h^{n-1}}{\tau_n} v_h \, dx + \sum_{K\in\mathcal{T}_{hn}} \int_K \nabla u_h^n \cdot \nabla v_h \, dx - \sum_{\Gamma\in\mathcal{F}_{hn}^{ID}} \int_\Gamma \langle \nabla u_h^n \cdot n\rangle [v_h] \, dS$$

$$+\theta \sum_{\Gamma\in\mathcal{F}_{hn}^{ID}} \int_\Gamma \langle \nabla v_h \cdot n\rangle [u_h^n] \, dS + \sum_{\Gamma\in\mathcal{F}_{hn}^{ID}} \int_\Gamma \sigma [u_h^n][v_h] \, dS = \int_\Omega f^n v_h \, dx$$

$$\text{for} \quad \forall\, v_h \in S_{h1}^n, \tag{50}$$

where $\theta = -1$, 1 and 0 is connected with the symetric form, the nonsymetric form and the incomplete form of discontinuous Galerkin method, respectively.

Definition 5 *Let* $\{u^n\}_{1\le n\le \bar N}$ *be the semi-discrete solution and* $\{u_h^n\}_{1\le n\le \bar N}$ *be the discrete solution of* (44)*.*
Then we set

$$\{e^n\}_{1\le n\le \bar N} = \{u^n - u_h^n\}_{1\le n\le \bar N}. \tag{51}$$

Lemma 1 *Let a triangulation* $\mathcal{T}_{hn}$ *satisfies* (6) *and* (7)*.*
Then there exists an operator $\Pi_{K,p} : H^1(K) \to P^p(K)$ *and a constant* $C_A > 0$ *such that*

$$|\Pi_{K,p}(v) - v|_{q,K} \le C_A h_K^{\mu-q} |v|_{\mu,K} \quad \forall\, v \in H^s(K) \quad \forall\, K \in \mathcal{T}_{hn}, \tag{52}$$

where $\mu =$ *min*$(p+1, s)$*,* $0 \le q \le s$ *and* $p, s \ge 1$ *are integers.*

Lemma 2 *Let a triangulation* $\mathcal{T}_{hn}$ *satisfies* (6) *and* (7). *The operator* $\Pi_{hp} : H^1(\Omega, \mathcal{T}_{hn}) \to S^n_{hp}$ *is defined by*

$$\Pi_{hp}|_K = \Pi_{K,p} \quad \forall\, K \in \mathcal{T}_{hn}, \tag{53}$$

and

$$|\Pi_{hp}(v) - v|_{H^q(\Omega,\mathcal{T}_{hn})} \leq C_A h_n^{\mu-q} |v|_{H^\mu(\Omega,\mathcal{T}_{hn})} \quad \forall\, v \in H^s(\Omega, \mathcal{T}_{hn}), \tag{54}$$

where $\mu =$ *min*$(p + 1, s)$, $0 \leq q \leq s$ *and* $p, s \geq 1$ *are integers.*

Definition 6 *The operator* $I^0_{hn} : H^1(\Omega, \widetilde{\mathcal{T}}_{hn}) \to S^n_{h1} \cap H^1_0(\Omega)$ *is defined by*

$$I^0_{hn}(v) = \mathcal{I}^0_{Os}(\Pi_{h1}(v)) \quad \forall\, v \in H^1(\Omega, \widetilde{\mathcal{T}}_{hn}), \tag{55}$$

where $\mathcal{I}^0_{Os}$ *is Oswald's operator corresponding to the homogeneous Dirichlet boundary condition.*

Definition 7 *Let $n \geq 1$. The local spatial error estimator is defined by*

$$\eta_K^n = h_K \left\| f^n - \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_K + h_K^{1/2} \|\nabla u_h^n \cdot n\|_{\partial K} + \|u_h^n\|_{H^{1/2}(\partial K)}$$
$$+ \sum_{\Gamma \in \mathcal{F}_{hn}^{ID} \cap \mathcal{F}_K} \left(h_\Gamma^{-1/2} \|[u_h^n]\|_\Gamma + h_\Gamma^{1/2} \|[u_h^n]\|_\Gamma \right), \qquad (56)$$

where $\mathcal{F}_K$ denotes the set of all edges and faces of a triangle and tetrahedra K, respectively.

Lemma 3 *The error e^n satisfies*

$$\sum_{K\in\widetilde{\mathcal{T}}_{hn}} \int_K \nabla e^n \cdot \nabla v_h \, dx = \int_\Omega \frac{e^{n-1} - e^n}{\tau_n} v_h \, dx$$

$$+ \theta \sum_{\Gamma\in\mathcal{F}^I_{hn}} \int_\Gamma \langle \nabla v_h \cdot n \rangle [u_h^n] \, dS \quad \forall\, v_h \in S^n_{h1} \cap H^1_0(\Omega). \quad (57)$$

Let us consider the splitting of the gradient of the error: $\nabla e^n = \nabla\phi^n + curl\,\chi^n$, then

$$\sum_{K\in\widetilde{\mathcal{T}}_{hn}} \|\nabla e^n\|^2_K = \underbrace{\sum_{K\in\widetilde{\mathcal{T}}_{hn}} \int_K \nabla e^n \cdot \nabla \phi^n \, dx}_{=:\psi} + \sum_{K\in\widetilde{\mathcal{T}}_{hn}} \int_K \nabla e^n curl\,\chi^n$$

(58)

Lemma 4 *The error e^n satisfies*

$$\sum_{K\in\widetilde{\mathcal{T}}_{hn}} \int_K \nabla e^n \cdot \nabla \phi \, dx = \int_\Omega (f^n - \frac{u^n - u^{n-1}}{\tau_n}) \phi \, dx$$

$$- \sum_{K\in\widetilde{\mathcal{T}}_{hn}} \int_{\partial K} \nabla u_h^n \cdot n \phi \, dS \quad \forall \, \phi \in H_0^1(\Omega). \quad (59)$$

$$\sum_{K\in\widetilde{\mathcal{T}}_{hn}} \int_K \nabla e^n curl \, \chi \, dx = - \sum_{K\in\widetilde{\mathcal{T}}_{hn}} \int_{\partial K} u_h^n curl \, \chi \cdot n \, dS$$

$$\forall \, \chi \in (H^1(\Omega))^k, \quad (60)$$

where $k = 1$ *for* $d = 2$ *and* $k = 3$ *for* $d = 3$*.*

$$
\begin{aligned}
\tau_n \psi = & \sum_{K \in \widetilde{\mathcal{T}}_{hn}} \int_K (e^n - e^{n-1})((e^n - \phi^n) - I^0_{hn}(e^n - \phi^n))\, dx \\
& + \sum_{K \in \widetilde{\mathcal{T}}_{hn}} \int_K (e^n - e^{n-1}) I^0_{hn}(e^n - \phi^n)\, dx \\
& + \sum_{K \in \widetilde{\mathcal{T}}_{hn}} \int_K e^n e^{n-1}\, dx - \sum_{K \in \widetilde{\mathcal{T}}_{hn}} \|e^n\|^2_K\, dx \\
& + \tau_n \sum_{K \in \widetilde{\mathcal{T}}_{hn}} \int_K (f^n - \frac{u^n_h - u^{n-1}_h}{\tau_n})(\phi^n - I^0_{hn}\phi^n)\, dx \\
& - \tau_n \sum_{K \in \widetilde{\mathcal{T}}_{hn}} \int_{\partial K} \nabla u^n_h \cdot n(\phi^n - I^0_{hn}\phi^n)\, dS \\
& + \tau_n \theta \sum_{\Gamma \in \mathcal{F}^I_{hn}} \int_\Gamma \langle \nabla I^0_{hn}\phi^n \cdot n \rangle [u^n_h]\, dS \quad .
\end{aligned}
\tag{61}
$$

Theorem 9 ***(Upper error bound)*** *Let* $\{u^n\}_{1\leq n\leq \bar{N}}$ *be the semi-discrete solution and* $\{u_h^n\}_{1\leq n\leq \bar{N}}$ *be the discrete solution of* (44). *Let* $1 \leq \bar{T} \leq \bar{N}$. *Then the error* e^n *defined in 5 satisfies*

$$\sum_{K\in\widetilde{\mathcal{T}}_{h\bar{T}}} \|e^{\bar{T}}\|_K^2 + \sum_{n=1}^{\bar{T}} \tau_n \sum_{K\in\widetilde{\mathcal{T}}_{hn}} \|\nabla e^n\|_K^2$$
$$\leq \sum_{K\in\widetilde{\mathcal{T}}_{h1}} \|e^0\|_K^2 + \sum_{n=1}^{\bar{T}} C_1(\eta^n)^2 \tag{62}$$
$$+ \sum_{n=1}^{\bar{T}} C_2(\eta^n)^2 \max\{h_n^2, \tau_n\},$$

where constants C_1, $C_2 > 0$.

- f_τ - piecewise constant and equal to $f(t_n)$ on each interval $(t_{n-1}, t_n]$, $1 \le n \le \bar{N}$
- $\{u^n\}_{0\le n\le \bar{N}} \dashrightarrow u_\tau$

$$u_\tau(t) = \frac{t_n - t}{\tau_n} u^{n-1} + \frac{t - t_{n-1}}{\tau_n} u^n \quad \forall t \in [t_{n-1}, t_n], 1 \le n \le \bar{N}. \tag{63}$$

Definition 8 *Let u be the weak solution and $\{u^n\}_{1\le n\le \bar{N}}$ be the semi-discrete solution of* (44)*. Then we set*

$$e_\tau = u - u_\tau, \tag{64}$$

where u_τ is precisely as in (63)*.*

Definition 9 *Let $1 \leq n \leq \bar{N}$. The time error indicator is defined by*

$$\eta_t^n = \tau_n^{1/2} |u_h^n - u_h^{n-1}|_{H^1(\Omega, \widetilde{\mathcal{T}}_{hn})}. \tag{65}$$

Theorem 10 ***(Time upper error bound)***
Let $\{u^n\}_{1\le n\le \bar N}$ *be the semi-discrete solution and* $\{u_h^n\}_{1\le n\le \bar N}$ *be the discrete solution of* (44). *Let* $1 \le \bar T \le \bar N$. *Then*

$$\|e_\tau(t_{\bar T})\|_\Omega^2 + \int_0^{t_{\bar T}} \|\nabla e_\tau(s)\|_\Omega^2\, ds \le 2\|f - f_\tau\|^2_{L^2(0,t_{\bar T};H^{-1}(\Omega))} + 2\sum_{n=1}^{\bar T}(\eta_t^n)^2 + 16\sum_{n=1}^{\bar T}\int_{t_{n-1}}^{t_n} |(u_\tau - u_{h\tau})(s)|^2_{H^1(\Omega,\widetilde{\mathcal{T}}_{hn})}\, ds. \tag{66}$$

Upper error bound - full discretization

Definition 10 *Let $1 \leq n \leq \bar{N}$. The full error indicator at time t_n is defined by*

$$E(t_n)^2 = \|u(t_n) - u^n\|_\Omega^2 + \|u^n - u_h^n\|_\Omega^2 + \left\| \frac{\partial e_\tau}{\partial t} \right\|_{L^2(0,t_n;H^{-1}(\Omega))}^2 + \left\| \frac{\partial (u_\tau - u_{h\tau})}{\partial t} \right\|_{L^2(0,t_n;H^{-1}(\Omega))}^2 + \int_0^{t_n} \|\nabla (u - u_\tau)(s)\|_\Omega^2 \, ds + \int_0^{t_n} |(u_\tau - u_{h\tau})(s)|_{H^1(\Omega,\widetilde{\mathcal{T}}_{hn})}^2 \, ds. \quad (67)$$

Upper error bound - full discretization

Theorem 11 ***(Full error bound)*** *Assume that the assumptions of Theorems 9 and 10 are satisfied. Let* $1 \leq \bar{T} \leq \bar{N}$*. Then*

$$E(t_{\bar{T}})^2 \leq 11 \sum_{n=1}^{\bar{T}} (\eta_t^n)^2 + 178\tau_1 \sum_{K \in \widetilde{\mathcal{T}}_{h1}} \left\| \nabla e^0 \right\|_K^2 + C_3 \sum_{K \in \widetilde{\mathcal{T}}_{h1}} \|e^0\|_K^2$$
$$+ 11 \|f - f_\tau\|^2_{L^2(0,t_{\bar{T}};H^{-1}(\Omega))} + C_4 \sum_{n=1}^{\bar{T}} (\eta^n)^2$$
$$+ C_5 \sum_{n=1}^{\bar{T}} (\eta^n)^2 \max\{h_n^2, \tau_n\}, \tag{68}$$

where constants C_3*,* C_4 *and* $C_5 > 0$*.*