New Approach to Automatic Adapticity Based on Fast Trial Refinements

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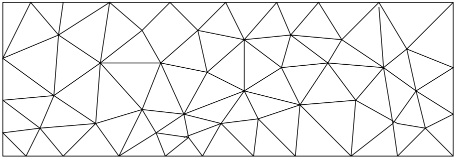
Automatic Adaptivity



Standard steps:

1. Solve on the coarse mesh.
2. Calculate a local error estimate for every element.
3. Identify elements with largest errors.
4. Refine those elements.
5. Go To 1.

Sometimes it Works

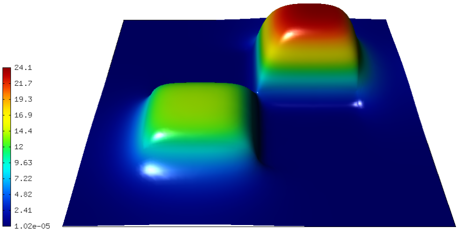


Figure: Stationary diffusion of neutrons.

Also Here It Works

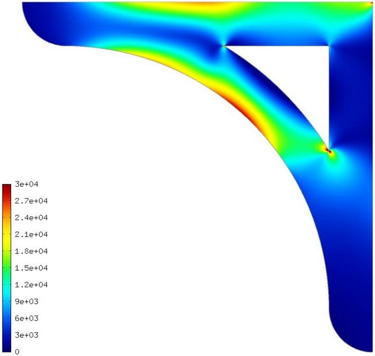


Figure: Small deformations of a bookshelf bracket.

But Not All Problems Are Elliptic

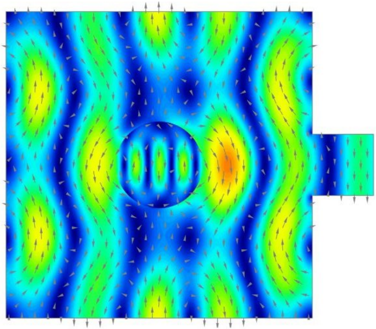


Figure: Microwave heating (Maxwell).

Navier-Stokes

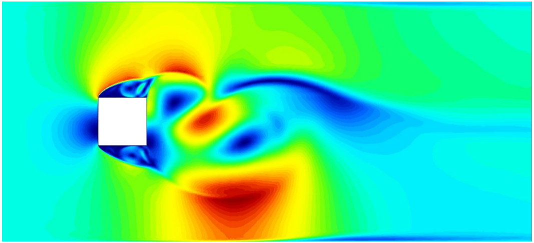


Figure: Incompressible flow.

Schrödinger

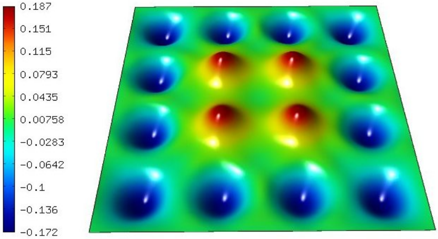
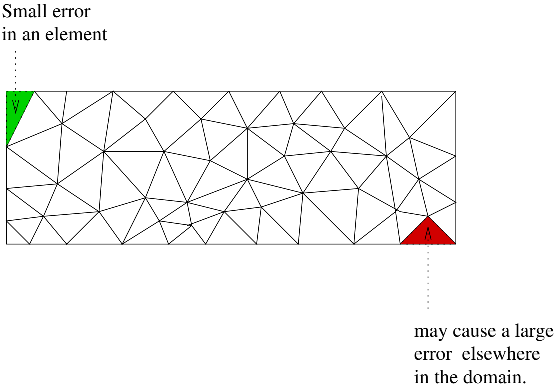


Figure: Oscillations in a Bose-Einstein condensate.

Non-Local Error



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The error will not decrease!

Unfortunately, most a-posteriori error estimates are *local*:

- Analytical error estimates.
- Solution of local Neumann or Dirichlet problems.
- Gradient recovery techniques.
- Etc.

Dual (Adjoint) Problem

1. Linear functional (quantity of interest): $L \in V'$
2. Continuous problem: $b(u, v) = l(v)$
3. Discrete problem: $b(u_{h,p}, v_{h,p}) = l(v_{h,p})$
4. Residual: $r \in V'$ such that $r(v) = l(v) - b(u_{h,p}, v)$
5. Relate the residual to the error in the quantity of interest:
   Find $G \in V''$ such that $G(r) = L(e_{h,p})$
6. By reflexivity: $G(r) = r(\hat{v})$ where $\hat{v} \in V$
7. But $L(e_{h,p}) = r(\hat{v}) = l(\hat{v}) - b(u_{h,p}, \hat{v}) = b(u, \hat{v}) - b(u_{h,p}, \hat{v}) = b(e_{h,p}, \hat{v})$
8. Dual problem:
   Find $\hat{v} \in V$ such that $b(e_{h,p}, \hat{v}) = L(e_{h,p})$ for all $e_{h,p} \in V$

Disadvantage: Linearity.

Looking for an Alternative

Requirements:

1. Has to identify sources of error.
2. Has to work well for nonlinear problems.
3. Has to be PDE independent.
4. Has to be fast.

Looking for an Alternative

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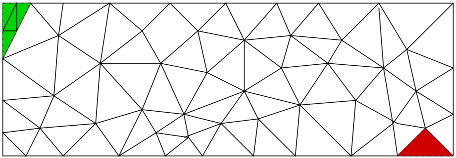
Fast Trial Refinements (FTR):



Local refinement + fast global solve.

Fast Global Solve

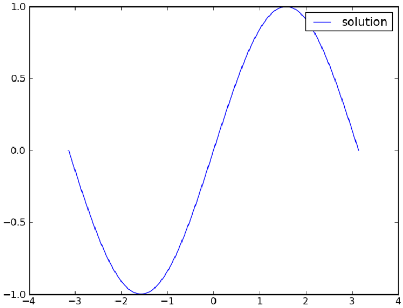
Jacobian-Free Newton-Krylov (JFNK) Method:

- Discrete problem (linear or nonlinear): $F(Y) = 0$.
- Newton's method:
$$J(Y^n)\delta Y^{n+1} = -F(Y^n).$$
- Finite difference approximation:
$$J(Y^n)v \approx \frac{F(Y^n + \epsilon v)}{\epsilon}.$$
- To be used with an iterative method such as GMRES.
- Coarse mesh approximation is an excellent initial guess.

No matrix assembled, no matrix problem solved.

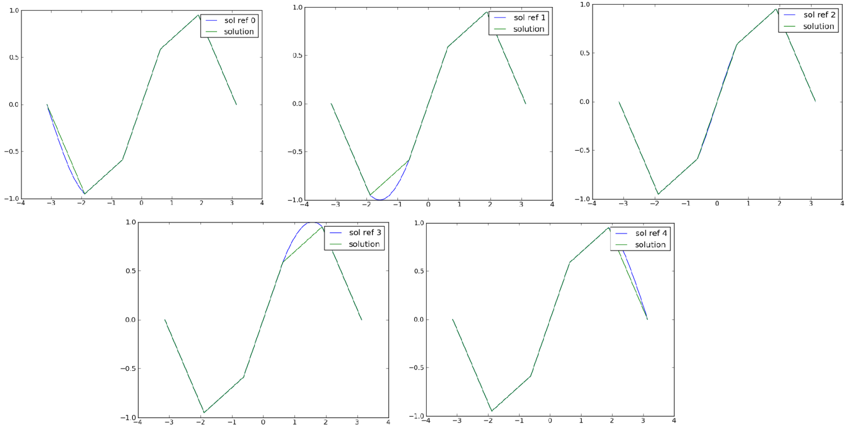
Example I

Equation $-u''(x) = f(x)$ in $(-\pi, \pi)$, zero Dirichlet BC.
Exact solution $u(x) = sin(x)$.
Computation goal: $L(u) = u(0.9\pi)$.



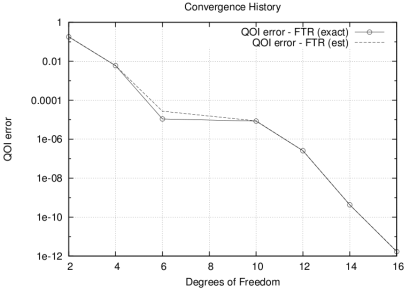
Example I

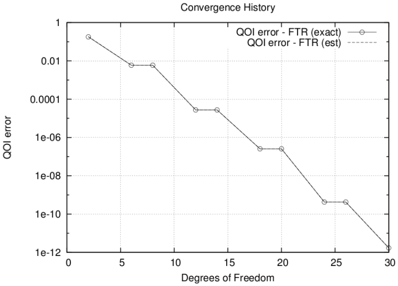
Reference solutions:



Example I

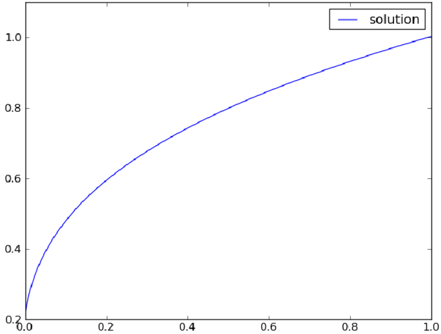
Convergence graphs:





Example II

Equation $-u''(x) = f(x)$ in $(0, 1)$, Dirichlet BC.
Exact solution $u(x) = x^{1/3+0.01}$.
Computation goal: $L(u) = u(0.9)$.



Example II

Convergence graphs: